(8 pages)

Reg. No. : ....

Code No. : 6377 **Sub. Code : ZMAM 31**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2022.

Third Semester

Mathematics — Core

ADVANCED ALGEBRA — I

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. If dim V=5 then dim Hom (V, V) + dim Hom (V, F) is

   (a) 50 (b) 25

   (c) 10 (d) 30

2. An R-module M is said to be cyclic if there is an element $m_0 \in m$ such that every $m \in M$ is of the form

   (a) $m = rm_0$ for some $r \in R$

   (b) $m = m_0^n$ for some integer n

   (c) $m = r + m_0$ for some $r \in R$

   (d) $m = rm_0$ for some $n \in M$

3. If V is finite dimensional over F and if $T \in A(V)$ is singular, then there exists an $S \neq 0$ in A(V) such that

   (a) ST=TS=1

   (b) ST=TS=0

   (c) $vS = 0$ for some $v \neq 0$ in V

   (d) ST=TS

4. If $vT = \lambda v$ the vT^k is

   (a) $(\lambda v)^k$ (b) λv^k

   (c) $\lambda^k v^k$ (d) $\lambda^k v$

5. If M, of dimension m, is cyclic w.r.t. T, then the dimension of MT^k is

   (a) $\frac{m}{k}$ (b) $m+k$

   (c) $m-k$ (d) m^k

6. Which one of the following is a Jordan block

   (a) $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

7. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}$ is the companion matrix of

   (a) $1+3x+3x^2$

   (b) $1+3x+3x^2+x^3$

   (c) $-1-3x-3x^2-x^3$

   (d) $-1-3x-3x^2+x^3+x^4$

8. Which one of the following is not true for all $A, B \in Fn$

   (a) $(A')' = A$

   (b) $(A+B)' = B' + A'$

   (c) $(AB)' = A'B'$

   (d) $(\lambda A')' = \lambda A'$ where $\lambda \in F$

9. The normal transformation N is unitary if and only if its characteristic roots are

   (a) real

   (b) complex number

   (c) all of absolute value 1

   (d) all equal to 1

10. The signature of the real quadratic form $x_1^2 + 2x_1x_2 + x^2$ is

    (a) 0 (b) -1

    (c) 1 (d) 2

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Define Ham(V, W). Introduce an addition and a scalar multiplication in Hom (V, W). Show that if $S, T \in Hom(V, W)$ than $S + T \in Hom(V, W)$.

    Or

(b) Let V be the set of all continuous complex - valued functions on [0, 1]. If $f(t), g(t) \in V$, define $(f|t|, g|t|) = \int f(t)\overline{g|t|}\,dt$. Prove that this defines an inner product on V.

12. (a) If $\lambda \in F$ is a characteristic root of $T \in A(V)$. Prove that for any polynomial of $q(x) \in F(x)$, $q(\lambda)$ is a characteristic root of q(T).

Or

(b) Let V be the vector space of all polynomials over F of degree 3 or less and let D be the differentiation operator defined on V. Find the matrix of D w.r.t. the basis

(i) $1, x, x^2, x^3$

(ii) $1, 1+x, 1+x^2, 1+x^3$

13. (a) If $W \subset V$ is invariant under T, prove that T induces a linear transformation $\overline{T}$ on V/W defined by $(V+W)\overline{T} = vT + w$.

Or

(b) If $T \in A(V)$ is nilpotent, prove that $\alpha_0 + \alpha_1 + T + ... + amT^m$, where the $\alpha_i \in F$, is invertible if $\alpha_o \neq 0$.

14. (a) If V is cyclic relative to T and if the minimal polynonomial of T in F[x] is $p(x)$, then prove that for some basis of V, the matrix of T is C(p(x)), the companion matrix of $p(x)$.

Or

(b) If A is invarible, prove that det $A \neq 0$, det $(A^{-1})=(\det A)^{-1}$ and det $(ABA^{-1})=$det B for all B.

15. (a) If $T \in A(V)$, prove that $T^* \in A(V)$ and $(T^*)^* = T$.

Or

(b) Let N be a normal transformation and suppose that λ and μ are two distinct characteristic roots of N. If V, W are in V are such that $VN = \lambda v$, $wN = \mu w$ prove that $(v, w) = 0$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If V and W are of dimensions m and n respectively exhibit a basis of Han (v, w) over F consisting of mn elements.

Or

(b) Let R be a Euclidean ring. Prove that any finitely generated R-module M is the direct sum of a finite number of cyclic submodules.

17. (a) For arbitrary algebras A with unit element over a field F, state and prove the analog of cayley's theorem for groups.

Or

(b) What relation, if any, must exist between characteristic vectors of T belonging to different characteristic roots? Explain your answer.

18. (a) If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, prove that T statisfies a polynomial of degree n over F.

Or

(b) Let $T \in A(v)$ and suppose that $p(x) = q_1(x)^{l1} q_2(x)^{l2} ... q_k(n)^{lk}$ in $F(x)$ is the minimal polynomial of T over F. For each $i = 1, 2......k$, define $V_i = \{v \in V \mid (V_{qi}(T))^{li} = 0\}$ prove that $v_i \neq 0$ for each i and $V = V_1 \oplus V_2 \oplus ... \oplus Vk$.

19. (a) Prove that the elements S and T in A(V) are similar in A(V) if and only if they have the same elementary divisions.

Or

(b) For $A, B \in Fn$ and $\lambda \in F$, Prove that

(i) $tr(\lambda A) = \lambda t_r A$

(ii) $tr(A+B) = trA + trB$.

(iii) $tr(AB) = tr(BA)$.

20. (a) Prove that to linear transformation T on V is unitary if and only if it takes an orthonomal basis of V into an orthonormal basis of V.

Or

(b) State and prove that sylvester's law of inertia.